A quantitative Hilbert's basis theorem and the constructive Krull dimension

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Hilbert's basis theorem (HBT) is an important topic in constructive algebra. In exploring constructive versions of HBT, several definitions of Noetherian rings have been considered, including Richman–Seidenberg Noetherian rings [Ric74, Sei74, MRR88]. Among them, Jacobsson and Löfwall's one [JL91] and Coquand and Persson's one [CP99] are (generalized) inductive definitions.

In this talk, we quantify the inductive definition and define α-Noetherian rings as follows, where α is an ordinal (as a Cantor Normal Form):

1. A list $[x_0, \ldots, x_{n-1}] \in \mathsf{List}\, A$ is called (-1)-*good* (or simply *good*) if $n \geq 1$ and $x_{n-1} \in \langle x_0, \ldots, x_{n-2} \rangle$.

2. A list $[x_0, \ldots, x_{n-1}] \in \mathsf{List}\, A$ is called α-*good* if for every $x \in A$, there exists $\beta \in [-1, \alpha)$ such that $[x_0, \ldots, x_{n-1}, x]$ is β-good.

3. A ring A is called α-*Noetherian* if the empty list $[] \in \mathsf{List}\, A$ is α-good.

Discrete fields are 1-Noetherian and $\mathbb{Z}$ is ω-Noetherian. We obtain the following constructive and quantitative version of HBT: if A is α-Noetherian, $A[X]$ is $(\omega \otimes \alpha)$-Noetherian, where $\otimes$ denotes the Hessenberg natural product.

Another important topic in constructive algebra is the Krull dimension. Lombardi [Lom02, Théorème 5] has obtained the following elementary characterization of the Krull dimension: $\mathsf{Kdim}\, A < n$ if and only if for all $x_0, \ldots, x_{n-1} \in A$, there exist $e_0, \ldots, e_{n-1} \geq 0$ such that

$$x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} \in \langle x_0^{e_0+1}, x_0^{e_0} x_1^{e_1+1}, \ldots, x_0^{e_0} \cdots x_{n-2}^{e_{n-2}} x_{n-1}^{e_{n-1}+1} \rangle.$$

This characterization can be used as a definition of the Krull dimension in constructive algebra [LQ15, Proposition XIII-2.8].

If a ring A is α-Noetherian for some $\alpha < \omega^n$, then $\mathsf{Kdim}\, A < n$ holds. Hence, the quantitative HBT gives an alternative constructive proof of $\mathsf{Kdim}\, K[X_0, \ldots, X_{n-1}] < 1 + n$ and $\mathsf{Kdim}\, \mathbb{Z}[X_0, \ldots, X_{n-1}] < 2 + n$, where K is a discrete field.

Our results have already been proved in classical mathematics [Gul73, Bro03]. Classically, a ring A is α-Noetherian for some $\alpha < \omega^n$ if and only if A is Noetherian and $\mathsf{Kdim}\, A < n$. We hope that the notion of α-Noetherian rings is useful for developing the constructive dimension theory of Noetherian rings.

References

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