A quantitative Hilbert's basis theorem and the constructive Krull dimension

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About me

Ryota Kuroki

Graduate student in mathematics at the University of Tokyo.

- My supervisor: Ryu Hasegawa
- Research interest: constructive algebra

What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants, such as Agda, Coq, Lean, ...

Proof of $\exists n \in \mathbb{N}.\ \varphi(n) \rightsquigarrow$ Algorithm to compute n s.t. $\varphi(n)$

Noetherian rings (non-constructive)

Definition 1

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots . \quad \exists n. \quad I_n = I_{n+1} = \cdots .$$

Example 1

1. All fields are Noetherian.
2. $\mathbb{Z}$ is Noetherian.
3. $\mathbb{Z}[X_0, X_1, \ldots]$ is not Noetherian.

$$0_{\mathbb{Z}} < \langle 12 \rangle < \langle 6 \rangle < \langle 3 \rangle < \langle 1 \rangle = \mathbb{Z}$$

$$0_{\mathbb{Z}[X_0, X_1, \ldots]} < \langle X_0 \rangle < \langle X_0, X_1 \rangle < \langle X_0, X_1, X_2 \rangle < \cdots$$

Noetherian rings (Richman–Seidenberg) (1/3)

Problems:

- There are mysterious ideals like $\{x \in \mathbb{Z} : (x = 0) \vee \varphi\}$.
- $\langle 2 \rangle \leq \langle 2 \rangle \leq \cdots$
  (is it $\langle 2 \rangle$ forever? or will it be $\mathbb{Z}$ at somewhere?).

There are several constructive definition of Noetherianity (Buriola, Schuster, and Blechschmidt [2023]).
Definition by Richman [1974] and Seidenberg [1974]:

Definition 2

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots \text{ (f.g.)} \quad \exists n. \quad I_n = I_{n+1}.$$

- If $I \leq \mathbb{Z}$ is f.g., we can compute $a \in \mathbb{Z}$ s.t. $I = \langle a \rangle$.
- We don't have to wait until I_n stabilizes.

Noetherian rings (Jacobsson–Löfwall) (2/3)

Generalized inductive definition by Jacobsson and Löfwall [1991]:

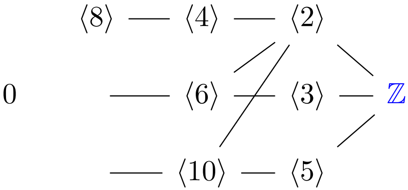
Definition 3

An ideal $I \leq A$ is *blocked* if

$$\forall x \in A.\ (x \notin I) \to (I + \langle x \rangle \text{ is blocked}).$$

A ring A is *Noetherian* if $0 \leq A$ is blocked.

(I prefer $\forall x \in A.\ (x \in I) \vee (I + \langle x \rangle$ is blocked).)



Noetherian rings (Coquand–Persson) (3/3)

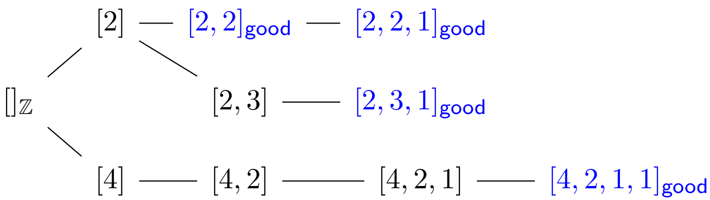
Generalized inductive definition by Coquand and Persson [1999]:

Definition 4

A list $[x_0, \ldots, x_{n-1}] \in \text{List } A$ is *good* if $\exists k. x_k \in \langle x_0, \ldots, x_{k-1} \rangle$.
A list $\sigma \in \text{List } A$ is *barred by good* if

$$(\sigma \text{ is good}) \vee (\forall x \in A.\ \sigma.x \text{ is barred by good}).$$

A ring is *Noetherian* if $[]$ is barred by good.



α-Noetherian rings

Definition 5

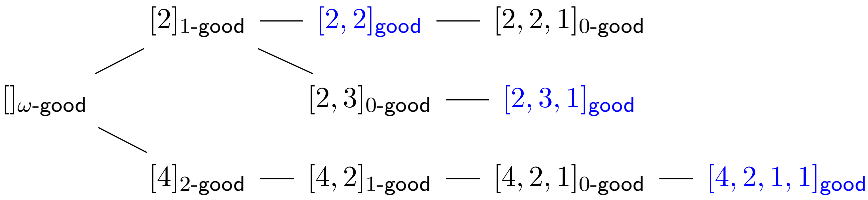
A list $[x_0, \ldots, x_{n-1}]$ is (-1)-*good* (or simply *good*) if

$$(n \geq 1) \wedge x_{n-1} \in \langle x_0, \ldots, x_{n-2} \rangle.$$

A list $\sigma \in \mathsf{List}\, A$ is α-*good* ($\alpha \in \mathsf{Ord}$) if

$$\forall x \in A.\ \exists \beta \in [-1, \alpha).\ \sigma.x \text{ is } \beta\text{-good}.$$

A ring is α-*Noetherian* if $[]$ is α-good.



Classically, the notion of α-Noetherian ring is introduced by Gulliksen [1973] as the length of Noetherian modules.

Examples of α-Noetherian rings

Example 2

1. Discrete fields $(\forall x.\ (x = 0) \vee (x \in K^{\times}))$ are 1-Noetherian
2. $\mathbb{Z}$ is ω-Noetherian.

More generally, we can define α-Euclidean rings and prove that they are α-Noetherian. (Classically, the notion of α-Euclidean ring is essentially introduced by Motzkin [1949].)

Definition 6

1. $x \in A$ is called (-1)-*Euclidean* if $x = 0$.
2. $x \in A$ is called α-*Euclidean* if for every $y \in A$, there exist $\beta \in [-1, \alpha)$ and β-Euclidean element $z \in A$ s.t. $z - y \in \langle x \rangle$.
3. A ring A is called α-*Euclidean* if for every $x \in A$, there exists $\beta \in [-1, \alpha)$ s.t. x is β-Euclidean.

Hilbert's basis theorem (HBT)

Theorem 7 (Classical HBT)

In classical mathematics, if A is Noetherian, then so is $A[X]$.

Theorem 8 (Coquand–Persson HBT)

If A is Coquand–Persson Noetherian, then so is $A[X]$.

There are also Richman–Seidenberg HBT and Jacobsson–Löfwall HBT, but those are theorems about

$$\text{Noetherianity} + (\text{some conditions like coherence}),$$

which are classically equivalent to Noetherianity.

Quantitative Hilbert's basis theorem (QHBT)

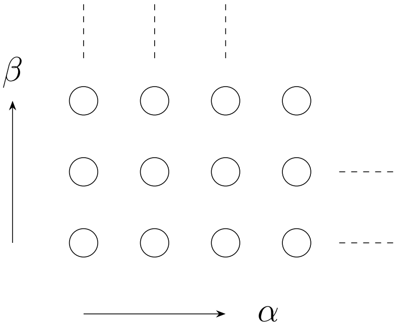
Theorem 9 (Kuroki [2025])

If A is α-Noetherian, then $A[X]$ is $(\omega \otimes \alpha)$-Noetherian.

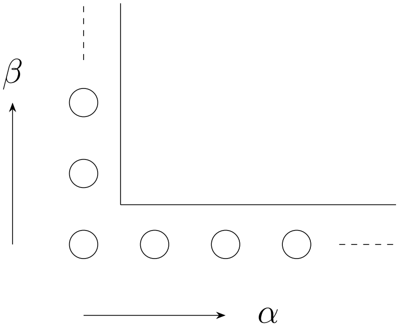
Classically, this is proved by Brookfield [2003].
To prove this theorem, we use a game called *(transfinite) chomp*.

Transfinite Chomp (1/4)

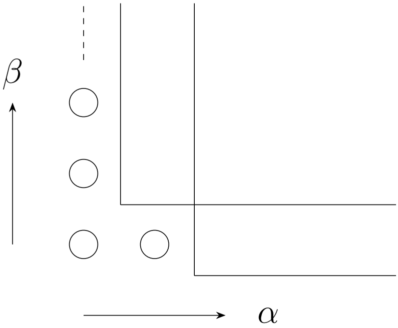
$(\alpha \times \beta)$-chomp (Huddleston and Shurman [2002])



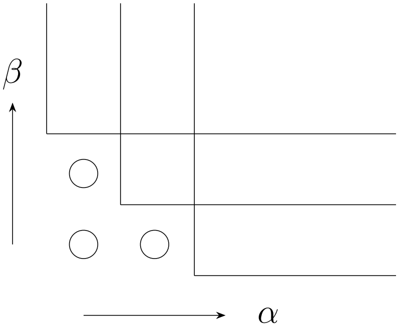
Transfinite Chomp (2/4)



Transfinite Chomp (3/4)



Transfinite Chomp (4/4)



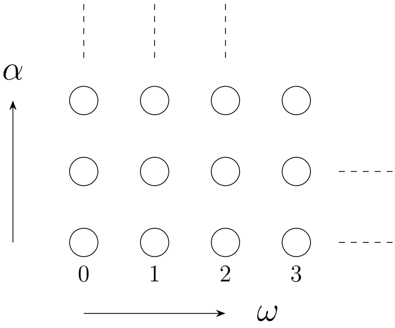
Transfinite Chomp (4/4)

The game ends in a finite number of steps. (Dickson's lemma)
Sketch of a proof (Huddleston and Shurman [2002]): We can assign an ordinal $\mathrm{size}\, P$ to each position P of the game. Every time you remove circles, the size decreases. □

The size of the initial position is $\alpha \otimes \beta$ (Hessenberg natural product).

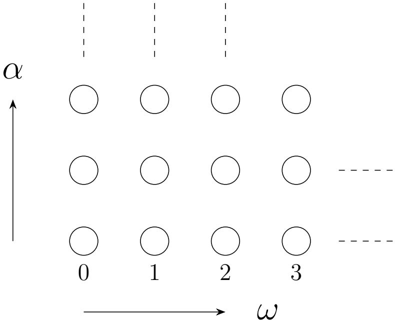
Proof of QHBT (1/10)

Assume that A is α-Noetherian (i.e., $[]_A$ is α-good).



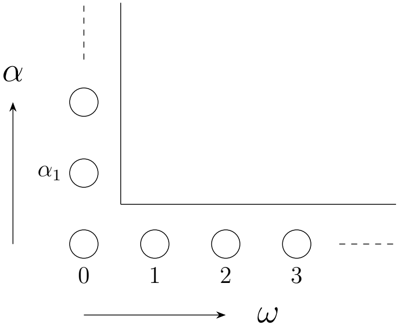
We prove that $[]$ is β_0-good, where
$\beta_0 :=$ (size of the above position) $= \omega \otimes \alpha$.
Suppose someone asks for $\beta_1 \in [-1, \beta_0)$ s.t.
$\sigma_1 := [a_1 X + a_0]$ is β_1-good.

Proof of QHBT (2/10): find β_1 s.t. σ_1 is β_1-good



We received $\sigma_1 = [a_1 X + a_0]$.
We ask for $\alpha_1 \in [-1, \alpha)$ s.t. $[a_1]_A$ is α_1-good.
Let's say $\alpha_1 = 1$.

Proof of QHBT (3/10): find β_1 s.t. σ_1 is β_1-good



We remove the top-right area from the point $(1, 1)$.

Meaning: $\exists f_0 \in \langle \sigma_1 \rangle.\ \deg f_0 = 1 \wedge [\mathrm{lc}\ f_0]_A$ is 1-good.

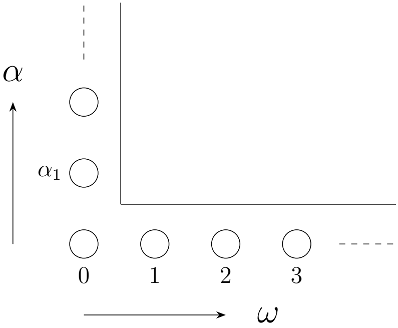
We prove that $\sigma_1 = [a_1 X + a_0]$ is β_1-good, where

$\beta_1 :=$ (size of the above position).

Suppose someone asks for $\beta_2 \in [-1, \beta_1)$ s.t.

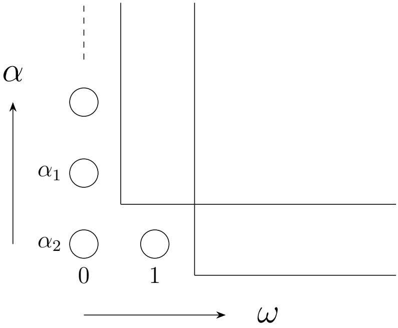
$\sigma_2 := [a_1 X + a_0, b_2 X^2 + b_1 X + b_0]$ is β_2-good.

Proof of QHBT (4/10): find β_2 s.t. σ_2 is β_2-good



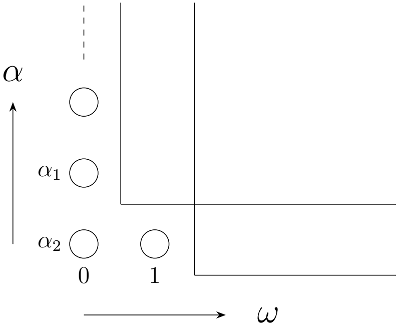
We received $\sigma_2 = [a_1 X + a_0, , b_2 X^2 + b_1 X + b_0]$.
We ask for $\alpha_2 \in [-1, \alpha_1)$ s.t. $[a_1, b_2]_A$ is α_2-good.
Let's say $\alpha_2 = 0$.

Proof of QHBT (5/10): find β_2 s.t. σ_2 is β_2-good



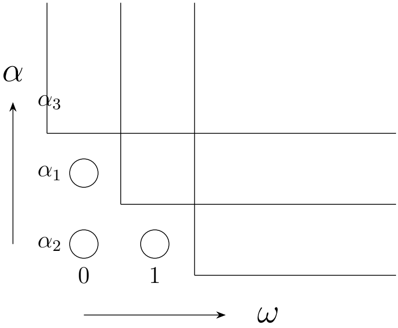
We remove the top-right area from the point $(2, 0)$.
Meaning: $\exists f_0, f_1 \in \langle \sigma_2 \rangle.\ \deg f_i = 2 \wedge [\mathrm{lc}\ f_0, \mathrm{lc}\ f_1]_A$ is 0-good.
We prove that $\sigma_2 = [a_1 X + a_0, b_2 X^2 + \cdots]$ is β_2-good, where
$\beta_2 :=$ (size of the above position).
Suppose someone asks for $\beta_3 \in [-1, \beta_1)$ s.t.
$\sigma_3 := [a_1 X + a_0, b_2 X^2 + \cdots, c_0]$ is β_3-good.

Proof of QHBT (6/10): find β_3 s.t. σ_3 is β_3-good



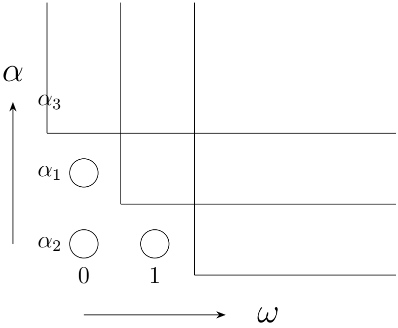
We received $\sigma_3 = [a_1X + a_0, , b_2X^2 + \cdots , c_0]$.
We ask for $\alpha_3 \in [-1, \alpha)$ s.t. $[c_0]_A$ is α_3-good.
Let's say $\alpha_3 = 2$.

Proof of QHBT (7/10): find β_3 s.t. σ_3 is β_3-good



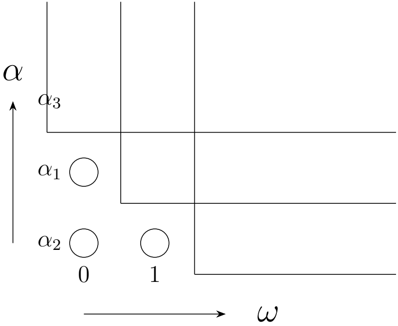
We remove the top-right area from the point $(2, 0)$.
Meaning: $\exists f_0 \in \langle \sigma_3 \rangle.\ \deg f_0 = 0 \wedge [\mathrm{lc}\ f_0]_A$ is 2-good.
We prove that $\sigma_3 = [a_1X + a_0, , b_2X^2 + \cdots , c_0]$ is β_3-good, where $\beta_3 :=$ (size of the above position).
Suppose someone asks for $\beta_4 \in [-1, \beta_3)$ s.t.
$\sigma_4 := [a_1X + a_0, \ldots, d_2X^2 + d_1X + d_0]$ is β_4-good.

Proof of QHBT (8/10): find β_4 s.t. σ_4 is β_4-good



We received $\sigma_4 = [a_1X + a_0, b_2X^2 + \cdots, c_0, d_2X^2 + d_1X + d_0]$.
We ask for $\alpha_4 \in [-1, \alpha_2)$ s.t. $[b_2, d_2]_A$ is α_4-good.
Then α_4 must be -1. Hence $d_2 \in \langle b_2 \rangle_A$. Hence
$\exists g \in A[X]. \deg g = 1 \wedge (g - (d_2X^2 + d_1X + d_0) \in \langle b_2X^2 + \cdots \rangle)$.

Proof of QHBT (9/10): find β_4 s.t. σ_4 is β_4-good



Write g as $d'_1 X + d'_0$.
We ask for $\alpha'_4 \in [-1, \alpha_1)$ s.t. $[a_1, d'_1]_A$ is α'_4-good...

Proof of QHBT (10/10)

Reduce the size of the position, reduce the degree of the polynomial at the end of the list, ...
By repeating this process, we can reduce the degree to -1. (When the size of the position reduces to 0, we have $1 \in \langle \sigma \rangle$.)

Hence $[]_{A[X]}$ is $(\omega \otimes \alpha)$-good.

Corollary 1

1. *If K is a discrete field, $K[X_0, \ldots, X_{n-1}]$ is ω^n-Noetherian.*
2. *$\mathbb{Z}[X_0, \ldots, X_{n-1}]$ is ω^{1+n}-Noetherian.*

Krull dimension (non-constructive)

Definition 10

We write Kdim $A < n$ if

$$\forall \mathfrak{p}_0 \leq \cdots \leq \mathfrak{p}_n.\ \exists k.\ \mathfrak{p}_k = \mathfrak{p}_{k+1}$$

Krull dimension (constructive)

Lombardi [2002] has found the following characterization:

Definition 11

We write Kdim $A < n$ if for every $x_0, \ldots, x_{n-1} \in A$, there exists $e_0, \ldots, e_{n-1} \geq 0$ such that

$$x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} \in \langle x_0^{e_0+1}, x_0^{e_0} x_1^{e_1+1}, \ldots, x_0^{e_0} \cdots x_{n-2}^{e_{n-2}} x_{n-1}^{e_{n-1}+1} \rangle.$$

As noted by [Lombardi, 2002, Proposition 5.2], this definition is closely related to the lexicographic order. For more results in this direction, see Kemper and Trung [2014], Kemper and Yengui [2020].

Example 3

1. Kdim $K < 1$ for a discrete field K.
2. Kdim $\mathbb{Z} < 2$.

α-Noetherianity and Krull dimension (1/2)

Theorem 12

Let $f : [0, \alpha) \to A$ be a function. If A is β-Noetherian for some $\beta < \alpha$, there exist $m \in \mathbb{N}$ and a strictly decreasing sequence $\alpha_0, \ldots, \alpha_{m-1} \in [0, \beta]$ s.t. $[f(\alpha_0), \ldots, f(\alpha_{m-1})]$ is good.

Proof.

Let $\alpha_0 := \beta$. Then $[f(\alpha_0)]$ is α_1-good for some $\alpha_1 \in [-1, \alpha_0)$.

1. If $\alpha_1 = -1$, then $[f(\alpha_0)]$ is good.
2. If $\alpha_1 \in [0, \alpha_0)$, then $[f(\alpha_0), f(\alpha_1)]$ is α_2-good for some $\alpha_2 \in [-1, \alpha_1)$...

□

α-Noetherianity and Krull dimension (2/2)

Theorem 13 (Classically proved by Gulliksen [1973])

If A is α-Noetherian for some $\alpha < \omega^n$, then Kdim $A < n$.

Proof.

Define $f : \omega^n \to A$ by $f(e_{n-1}, \ldots, e_1, e_0) := x_0^{e_0} \cdots x_{n-1}^{e_{n-1}}$. □

Corollary 2 (Lombardi [2002], Lombardi and Quitté [2015])

1. *If K is a discrete field,* Kdim $K[X_0, \ldots, X_{n-1}] < 1 + n$.
2. Kdim $\mathbb{Z}[X_0, \ldots, X_{n-1}] < 2 + n$.

Summary and future work

The notion of α-Noetherian ring works well with Krull dimension.

Future work: Constructive dimension theory of Noetherian rings

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