

Entailment relations for the constructive theory of free modules

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23 December 2025

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About me

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What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants.

Proof of $\exists n \in \mathbb{N}. \varphi(n) \rightsquigarrow$ Algorithm to compute n s.t. $\varphi(n)$

- 1 Classical proof
- 2 Entailment relations
- 3 Conservative extension
- 4 Summary

$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$, classically

Theorem 1

If $e_s =_{A \oplus S} e_t$, then $s =_S t$ or $1 =_A 0$.

Classical proof.

By LEM, we can define $f : S \rightarrow A^S$ by

$$f(s)(t) := \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

This map induces a homomorphism $\tilde{f} : A^{\oplus S} \rightarrow A^S$. Then we have $f(s)(t) = \tilde{f}(e_s)(t) = \tilde{f}(e_t)(t) = f(t)(t) = 1$.

- If $s = t$, we have $s = t$.
- If $s \neq t$, we have $0 = f(s)(t) = 1$.

By LEM, we have $s = t$ or $1 = 0$.



Constructivizing the proof

Some constructive proofs are collected in the following MathOverflow discussion:

- *Constructively, is the unit of the “free abelian group” monad on sets injective?* <https://mathoverflow.net/q/302516>.

According to Blechschmidt, we can apply the baby version of Barr’s theorem (Friedman’s translation).

In this talk, I will show a simple constructivization method using entailment relations. (I believe that this method is essentially the same as baby Barr.)

Entailment relations

Definition 2 (Lorenzen [1951], Scott [1971])

A binary relation \vdash on the set of finite subsets of P is called an entailment relation on P if \vdash satisfies the following conditions:

$$\frac{}{\varphi \vdash \varphi} \ , \quad \frac{U \vdash V}{U, U' \vdash V, V'} \ , \quad \frac{U, \varphi \vdash V \quad U' \vdash \varphi, V'}{U, U' \vdash V, V'} \ .$$

The application of entailment relations to constructive algebra dates back to Cederquist and Coquand [2000], Coquand and Persson [2001] (The application of distributive lattices dates back to Joyal [1975]).

Examples: \vdash_S, \vdash_A

We define an entailment relation \vdash_S on $\{\text{Eq}_S(s, t) : s, t \in S\}$.

Axioms of \vdash_S :

$$\begin{aligned} &\vdash \text{Eq}_S(s, s), \quad \text{Eq}_S(s, t), \text{Eq}_S(t, u) \vdash \text{Eq}_S(s, u), \\ &\text{Eq}_S(s, t) \vdash \text{Eq}_S(t, s). \end{aligned}$$

Proposition 1

- $U \vdash_S \varphi_0, \dots, \varphi_{n-1} \iff \exists k. U \vdash_S \varphi_k.$
- $U \vdash_S \text{Eq}_S(s, t) \iff s \sim_U t.$

Corollary 1

$$\vdash_S \text{Eq}_S(s, t) \iff s = t.$$

We define \vdash_A similarly.

Example: $\vdash_{S,A}$

We define an entailment relation $\vdash_{S,A}$ on $\{\text{Eq}_S(s, t) : s, t \in S\} \cup \{\text{Eq}_A(a, b) : a, b \in A\}$.

Axioms of $\vdash_{S,A}$: axioms of \vdash_S and \vdash_A .

Proposition 2

$$U_S, U_A \vdash_{S,A} V_S, V_A \iff (U_S \vdash_S V_S \text{ or } U_A \vdash_A V_A).$$

Corollary 2

- $\vdash_{S,A} \text{Eq}_S(s, t) \iff s = t.$
- $\vdash_{S,A} \text{Eq}_A(a, b) \iff a = b.$
- $\vdash_{S,A} \text{Eq}_S(s, t), \text{Eq}_A(1, 0) \iff (s = t \text{ or } 1 =_A 0).$

We will extend $\vdash_{S,A}$ conservatively.

Adjoining \neg

Theorem 3 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin $\{\neg\varphi : \varphi \in P\}$ to an entailment relation with the following axioms:

$$\varphi, \neg\varphi \vdash, \quad \vdash \varphi, \neg\varphi.$$

Proof sketch.

By induction,

$$U_0, \neg U_1 \vdash \neg V_0, \neg V_1 \implies U_0, V_1 \vdash V_0, U_1 \quad (U_k, V_k: \text{negation-free}).$$



Key point: Adjoining f

Proposition 3

We can conservatively adjoin a function symbol f to $\vdash_{S,A,\neg}$ with the following axioms:

$$\begin{aligned} \text{Eq}_S(s, t) \vdash \text{Eq}_A(f(s, t), 1), \quad \neg \text{Eq}_S(s, t) \vdash \text{Eq}_A(f(s, t), 0), \\ \vdash \text{Eq}_A(\alpha, \alpha), \quad \text{Eq}_A(\alpha, \beta), \text{Eq}_A(\beta, \gamma) \vdash \text{Eq}_A(\alpha, \gamma), \quad \dots \\ (\alpha, \beta, \gamma \text{ may contain } f). \end{aligned}$$

We use the fundamental theorem of entailment relation to prove this.

Fundamental theorem of entailment relation

Theorem 4 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin connectives $\top, \wedge, \perp, \vee$ with the following axioms:

$$\begin{aligned} &\top \vdash, \quad \varphi, \psi \vdash \varphi \wedge \psi, \quad \varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi, \\ &\perp \vdash, \quad \varphi \vdash \varphi \vee \psi, \quad \psi \vdash \varphi \vee \psi, \quad \varphi \vee \psi \vdash \varphi, \psi. \end{aligned}$$

Proof sketch.

$$\begin{aligned} \llbracket U; V \rrbracket &:= U \vdash V \quad (U, V: (\top, \wedge, \perp, \vee)\text{-free}), \\ \llbracket U, \top; V \rrbracket &:= \llbracket U; V \rrbracket, \\ \llbracket U, \varphi \wedge \psi; V \rrbracket &:= \llbracket U, \varphi, \psi; V \rrbracket, \\ \llbracket U, \perp; V \rrbracket &:= \text{true}, \\ \llbracket U, \varphi \vee \psi; V \rrbracket &:= \llbracket U, \varphi; V \rrbracket \text{ and } \llbracket U, \psi; V \rrbracket, \dots \end{aligned}$$

Then, prove $U \vdash_{\text{DLat}} V \Rightarrow \llbracket U; V \rrbracket$ by induction.



Adjoining f (proof)

Proposition 3

We can conservatively adjoin a function symbol f to $\vdash_{S,A,\neg}$ with the following axioms:

$$\begin{aligned} \text{Eq}_S(s, t) \vdash \text{Eq}_A(f(s, t), 1), \quad \neg \text{Eq}_S(s, t) \vdash \text{Eq}_A(f(s, t), 0), \\ \vdash \text{Eq}_A(a, a), \quad \text{Eq}_A(a, b), \text{Eq}_A(b, c) \vdash \text{Eq}_A(a, c), \quad \dots \\ (a, b, c \text{ may contain } f). \end{aligned}$$

Proof sketch.

$$\begin{aligned} & \llbracket \varphi(f(s_0, t_0), \dots, f(s_{n-1}, t_{n-1})) \rrbracket \\ & := \bigvee_{i_0, \dots, i_{n-1} \in \{0,1\}} (\neg^{i_0} \text{Eq}_S(s_0, t_0) \wedge \dots \wedge \neg^{i_{n-1}} \text{Eq}_S(s_{n-1}, t_{n-1}) \\ & \quad \wedge \varphi(1 - i_0, \dots, 1 - i_{n-1})). \end{aligned}$$

Then, prove $U \vdash_{S,A,\neg,f} V \Rightarrow \llbracket U \rrbracket \vdash_{S,A,\neg,\text{DLat}} \llbracket V \rrbracket$ by induction. \square

Adjoining $+'_A, \cdot'_A$

Proposition 5

We can conservatively adjoin a function symbol $+', \cdot'$ to $\vdash_{S,A,\neg,f}$ with the following axioms:

$$\begin{aligned} &\vdash \text{Eq}_A(\alpha, \alpha), \quad \text{Eq}_A(\alpha, \beta), \text{Eq}_A(\beta, \gamma) \vdash \text{Eq}_A(\alpha, \gamma), \dots, \\ &\quad \text{Eq}_A(\alpha, \beta) \vdash \text{Eq}_A(\alpha +'_A \gamma, \beta +'_A \gamma), \dots, \\ &\vdash \text{Eq}_A(0 +'_A \alpha, \alpha), \vdash \text{Eq}_A(1 \cdot'_A \alpha, \alpha), \dots \quad (\alpha, \beta, \gamma \text{ may contain } f, +', \cdot'), \\ &\quad \vdash \text{Eq}_A(a +'_A b, a + b), \quad \vdash \text{Eq}_A(a \cdot'_A b, ab) \quad (a, b \in A). \end{aligned}$$

Proof sketch.

$$\begin{aligned} &\llbracket \varphi(f(s_0, t_0), \dots, f(s_{n-1}, t_{n-1})) \rrbracket \\ &:= \bigvee_{i_0, \dots, i_{n-1} \in \{0,1\}} (\neg^{i_0} \text{Eq}_S(s_0, t_0) \wedge \dots \wedge \neg^{i_{n-1}} \text{Eq}_S(s_{n-1}, t_{n-1}) \\ &\quad \wedge \varphi(1 - i_0, \dots, 1 - i_{n-1})). \quad (+', \cdot' \rightsquigarrow +, \cdot) \quad \square \end{aligned}$$

Adjoining \tilde{f} (1/2)

Let A_S be the set generated by the following constructors:

$$\frac{s \in S}{e_s \in A_S} \ , \quad \frac{}{0 \in A_S} \ , \quad \frac{x, y \in A_S}{x + y \in A_S} \ , \quad \frac{a \in A, x \in A_S}{ax \in A_S} \ .$$

Note that $A^{\oplus S} \cong A_S / \sim$, where \sim is generated by $0 + x \sim x, \dots$

We adjoin elements of the form $\tilde{f}(x, t)$ ($x \in A_S, t \in S$) and extend $+', \cdot'$ to elements containing \tilde{f} with the following axioms:

$$\begin{aligned} &\vdash \text{Eq}_A(\tilde{f}(e_s, t), f(s, t)), \quad \vdash \text{Eq}_A(\tilde{f}(0, t), 0_A), \\ &\vdash \text{Eq}_A(\tilde{f}(x + y, t), \tilde{f}(x, t) +' \tilde{f}(y, t)), \quad \vdash \text{Eq}_A(\tilde{f}(ax, t), a \cdot' \tilde{f}(x, t)), \\ &\vdash \text{Eq}_A(0 +' \alpha, \alpha), \dots \quad (\alpha \text{ may contain } \tilde{f}) \end{aligned}$$

Adjoining \tilde{f} (2/2)

Proposition 6

We can conservatively adjoin \tilde{f} to $\vdash_{S,A,\neg,f,+',\cdot'}$ (and extend $+',\cdot'$).

Proof sketch.

$$\llbracket \tilde{f}(e_s, t) \rrbracket := f(s, t)$$

$$\llbracket \tilde{f}(0, t) \rrbracket := 0$$

$$\llbracket \tilde{f}(x + y, t) \rrbracket := \llbracket \tilde{f}(x, t) \rrbracket +' \llbracket \tilde{f}(y, t) \rrbracket$$

$$\llbracket \tilde{f}(ax, t) \rrbracket := a \cdot' \llbracket \tilde{f}(x, t) \rrbracket.$$



$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$, constructively

Proposition 7

If $x =_{A \oplus S} y$, then $\vdash_{S, A, \neg, f, +', \cdot', \tilde{f}} \text{Eq}_A(\tilde{f}(x, t), \tilde{f}(y, t))$.

Proof Sketch.

Note that $A^{\oplus S} \cong A_S / \sim$, where \sim is the equivalence relation generated by $0 + x \sim x, \dots$



Constructive proof of Theorem 1.

Suppose $e_s =_{A \oplus S} e_t$. Then $\vdash_{S, A, \neg, f, +', \cdot', \tilde{f}} \text{Eq}_A(\tilde{f}(e_s, t), \tilde{f}(e_t, t))$. By simulating the classical proof on the entailment relation, we have

$$\vdash_{S, A, \neg, f, +', \cdot', \tilde{f}} \text{Eq}_S(s, t), \text{Eq}_A(1, 0)$$

By the conservativity, we have $\vdash_{S, A} \text{Eq}_S(s, t), \text{Eq}_A(1, 0)$. Hence $(s =_S t) \vee (1 =_A 0)$ holds.



Summary and future work

Using entailment relations, we can deal with *ideal objects* such as f and \tilde{f} .

Future work: Combine this method with other applications of entailment relations.

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