A quantitative Hilbert's basis theorem and the constructive Krull dimension

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17 October 2025

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About me

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What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants.

Proof of $\exists n \in \mathbb{N}.\ \varphi(n) \rightsquigarrow$ Algorithm to compute n s.t. $\varphi(n)$

Noetherian rings (non-constructive)

Definition 1

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots . \quad \exists n. \quad I_n = I_{n+1} = \cdots .$$

Example 1

1. All fields are Noetherian.
2. $\mathbb{Z}$ is Noetherian.
3. $\mathbb{Z}[X_0, X_1, \ldots]$ is not Noetherian.

$$0_{\mathbb{Z}} < \langle 12 \rangle < \langle 6 \rangle < \langle 3 \rangle < \langle 1 \rangle = \mathbb{Z}$$

$$0_{\mathbb{Z}[X_0,X_1,\ldots]} < \langle X_0 \rangle < \langle X_0, X_1 \rangle < \langle X_0, X_1, X_2 \rangle < \cdots$$

Noetherian rings (Richman–Seidenberg)

Problems:

- There are mysterious ideals like $\{x \in \mathbb{Z} : (x = 0) \vee \varphi\}$.
- $\langle 2 \rangle \leq \langle 2 \rangle \leq \cdots$
  (is it $\langle 2 \rangle$ forever? or will it be $\mathbb{Z}$ at somewhere?).

There are several constructive definition of Noetherianity (Buriola, Schuster, and Blechschmidt [2023]).
Definition by Richman [1974] and Seidenberg [1974]:

Definition 2

A ring A is *Noetherian* if

$$\forall I_0 \leq I_1 \leq \cdots \text{ (f.g.)} \quad \exists n. \quad I_n = I_{n+1}.$$

- If $I \leq \mathbb{Z}$ is f.g., we can compute $a \in \mathbb{Z}$ s.t. $I = \langle a \rangle$.
- We don't have to wait until I_n stabilizes.

Noetherian rings (Jacobsson–Löfwall)

Generalized inductive definition by Jacobsson and Löfwall [1991]:

Definition 3

An ideal $I \leq A$ is *blocked* if

$$\forall x \in A.\ (x \notin I) \to (I + \langle x \rangle \text{ is blocked}).$$

A ring A is *Noetherian* if $0 \leq A$ is blocked.

(I prefer $\forall x \in A.\ (x \in I) \vee (I + \langle x \rangle \text{ is blocked})$.)

$$\begin{tikzcd}[arrows=dash]
 & \langle 8 \rangle \arrow[r] & \langle 4 \rangle \arrow[r] & \langle 2 \rangle \arrow[dr] & \\
0 \arrow[r] \arrow[dr] & \langle 6 \rangle \arrow[r] \arrow[ur] & \langle 3 \rangle \arrow[r] & \mathbb{Z} \\
 & \langle 10 \rangle \arrow[r] \arrow[uur] & \langle 5 \rangle \arrow[ur] &
\end{tikzcd}$$

Noetherian rings (Coquand–Persson)

Generalized inductive definition by Coquand and Persson [1999]:

Definition 4

A list $[x_0, \ldots, x_{n-1}] \in \text{List } A$ is *good* if $\exists k. x_k \in \langle x_0, \ldots, x_{k-1} \rangle$.
A list $\sigma \in \text{List } A$ is *barred by good* if

$$(\sigma \text{ is good}) \vee (\forall x \in A.\ \sigma.x \text{ is barred by good}).$$

A ring is *Noetherian* if $[]$ is barred by good.

$$\begin{tikzcd}[arrows=dash]
 & {[2]} \arrow[r] \arrow[dr] & {[2,2]_{\text{good}}} \arrow[r] & {[2,2,1]_{\text{good}}} & \\
{[]_{\mathbb{Z}}} \arrow[ur] \arrow[dr] & & {[2,3]} \arrow[r] & {[2,3,1]_{\text{good}}} & \\
 & {[4]} \arrow[r] & {[4,2]} \arrow[r] & {[4,2,1]} \arrow[r] & {[4,2,1,1]_{\text{good}}}
\end{tikzcd}$$

α-Noetherian rings

Definition 5

A list $[x_0, \ldots, x_{n-1}]$ is (-1)-*good* (or simply *good*) if

$$(n \geq 1) \wedge x_{n-1} \in \langle x_0, \ldots, x_{n-2} \rangle.$$

A list $\sigma \in \mathsf{List}\, A$ is α-*good* ($\alpha \in \mathsf{Ord}$) if

$$\forall x \in A.\ \exists \beta \in [-1, \alpha).\ \sigma.x \text{ is } \beta\text{-good}.$$

A ring is α-*Noetherian* if $[]$ is α-good.

$$\begin{tikzcd}[column sep=small]
 & {[2]_{1\text{-good}}} \arrow[r, dash] \arrow[dr, dash] & {[2,2]_{\text{good}}} \arrow[r, dash] & {[2,2,1]_{0\text{-good}}} & \\
{[]_{\omega\text{-good}}} \arrow[ur, dash] \arrow[dr, dash] & & {[2,3]_{0\text{-good}}} \arrow[r, dash] & {[2,3,1]_{\text{good}}} & \\
 & {[4]_{2\text{-good}}} \arrow[r, dash] & {[4,2]_{1\text{-good}}} \arrow[r, dash] & {[4,2,1]_{0\text{-good}}} \arrow[r, dash] & {[4,2,1,1]_{\text{good}}}
\end{tikzcd}$$

Classically, the notion of α-Noetherian ring is introduced by Gulliksen [1973] as the length of Noetherian modules.

Examples of α-Noetherian rings

Example 2

1. Discrete fields $(\forall x.\ (x = 0) \vee (x \in K^{\times}))$ are 1-Noetherian
2. $\mathbb{Z}$ is ω-Noetherian.

More generally, we can define α-Euclidean rings and prove that they are α-Noetherian. (Classically, the notion of α-Euclidean ring is essentially introduced by Motzkin [1949].)

Definition 6

1. $x \in A$ is called (-1)-*Euclidean* if $x = 0$.
2. $x \in A$ is called α-*Euclidean* if for every $y \in A$, there exist $\beta \in [-1, \alpha)$ and β-Euclidean element $z \in A$ s.t. $z - y \in \langle x \rangle$.
3. A ring A is called α-*Euclidean* if for every $x \in A$, there exists $\beta \in [-1, \alpha)$ s.t. x is β-Euclidean.

Hilbert's basis theorem (HBT)

Theorem 7 (Classical HBT)

In classical mathematics, if A is Noetherian, then so is $A[X]$.

Theorem 8 (Coquand and Persson [1999])

If A is Coquand–Persson Noetherian, then so is $A[X]$.

There are also Richman–Seidenberg HBT and Jacobsson–Löfwall HBT, but those are theorems about

$$\text{Noetherianity} + (\text{some conditions like coherence}),$$

which are classically equivalent to Noetherianity.

Quantitative Hilbert's basis theorem (QHBT)

Theorem 9 (Kuroki [2025])

If A is α-Noetherian, then $A[X]$ is $(\omega \otimes \alpha)$-Noetherian.

Classically, this is proved by Brookfield [2003].

Corollary 1

1. *If K is a discrete field, $K[X_0, \ldots, X_{n-1}]$ is ω^n-Noetherian.*
2. $\mathbb{Z}[X_0, \ldots, X_{n-1}]$ *is ω^{1+n}-Noetherian.*

Krull dimension (non-constructive)

Definition 10

We write $\operatorname{Kdim} A < n$ if

$$\forall \mathfrak{p}_0 \leq \cdots \leq \mathfrak{p}_n.\ \exists k.\ \mathfrak{p}_k = \mathfrak{p}_{k+1}$$

Krull dimension (constructive)

Lombardi [2002] has found the following characterization:

Definition 11

We write $\mathrm{Kdim}\, A < n$ if for every $x_0, \ldots, x_{n-1} \in A$, there exists $e_0, \ldots, e_{n-1} \geq 0$ such that

$$x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} \in \langle x_0^{e_0+1},\ x_0^{e_0} x_1^{e_1+1}, \ldots,\ x_0^{e_0} \cdots x_{n-2}^{e_{n-2}} x_{n-1}^{e_{n-1}+1} \rangle.$$

As noted by [Lombardi, 2002, Proposition 5.2], this definition is closely related to the lexicographic order. For more results in this direction, see Kemper and Trung [2014], Kemper and Yengui [2020].

Example 3

1. $\mathrm{Kdim}\, K < 1$ for a discrete field K.
2. $\mathrm{Kdim}\, \mathbb{Z} < 2$.

α-Noetherianity and Krull dimension (1/2)

Theorem 12 (Kuroki [2025])

Let $f : [0, \alpha) \to A$ be a function. If A is β-Noetherian for some $\beta < \alpha$, there exist $m \in \mathbb{N}$ and a strictly decreasing sequence $\alpha_0, \ldots, \alpha_{m-1} \in [0, \beta]$ s.t. $[f(\alpha_0), \ldots, f(\alpha_{m-1})]$ is good.

Proof.

Let $\alpha_0 := \beta$. Then $[f(\alpha_0)]$ is α_1-good for some $\alpha_1 \in [-1, \alpha_0)$.

1. If $\alpha_1 = -1$, then $[f(\alpha_0)]$ is good.
2. If $\alpha_1 \in [0, \alpha_0)$, then $[f(\alpha_0), f(\alpha_1)]$ is α_2-good for some $\alpha_2 \in [-1, \alpha_1)$...

□

α-Noetherianity and Krull dimension (2/2)

Theorem 13 (Kuroki [2025], Classically proved by Gulliksen [1973])

If A is α-Noetherian for some $\alpha < \omega^n$, then Kdim $A < n$.

Proof.

Define $f : \omega^n \to A$ by $f(e_{n-1}, \ldots, e_1, e_0) := x_0^{e_0} \cdots x_{n-1}^{e_{n-1}}$. $\square$

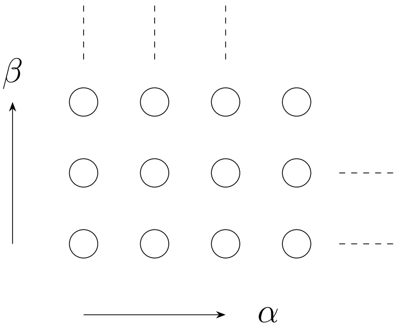
Corollary 2 (Lombardi [2002], Lombardi and Quitté [2015])

1. *If K is a discrete field,* Kdim $K[X_0, \ldots, X_{n-1}] < 1 + n$.
2. Kdim $\mathbb{Z}[X_0, \ldots, X_{n-1}] < 2 + n$.

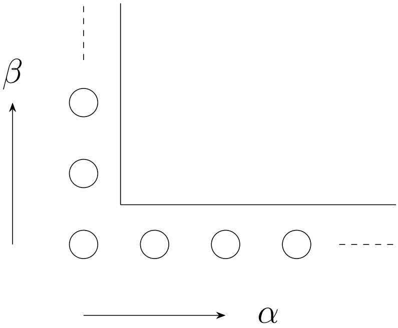
Transfinite Chomp (1/5)

To prove QHBT, we use a game called *(transfinite) chomp* (Huddleston and Shurman [2002]).

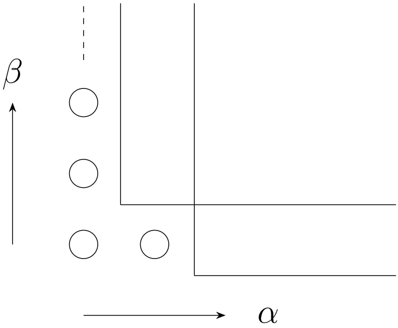
$(\alpha \times \beta)$-chomp:



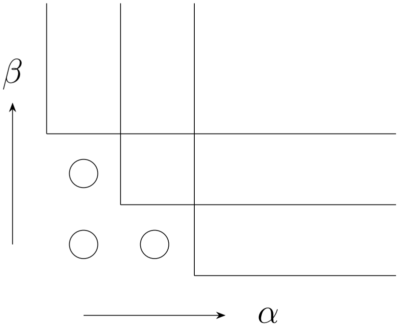
Transfinite Chomp (2/5)



Transfinite Chomp (3/5)



Transfinite Chomp (4/5)



Transfinite Chomp (5/5)

The game ends in a finite number of steps. (Dickson's lemma)
Sketch of a proof (Huddleston and Shurman [2002]): We can assign an ordinal $\mathrm{size}\,P$ to each position P of the game. Every time you remove circles, the size decreases. $\square$

The size of the initial position is $\alpha \otimes \beta$ (Hessenberg natural product).

Summary and future work

The notion of α-Noetherian ring works well with Krull dimension.

Future work: Constructive dimension theory of Noetherian rings

References I

Gary Brookfield. The length of Noetherian polynomial rings. *Comm. Algebra*, 31(11): 5591–5607, 2003. ISSN 0092-7872,1532-4125. doi: 10.1081/AGB-120023976. URL https://doi.org/10.1081/AGB-120023976.

Gabriele Buriola, Peter Schuster, and Ingo Blechschmidt. A constructive picture of Noetherian conditions and well quasi-orders. In *Unity of logic and computation*, volume 13967 of *Lecture Notes in Comput. Sci.*, pages 50–62. Springer, Cham, 2023. ISBN 978-3-031-36977-3; 978-3-031-36978-0. doi: 10.1007/978-3-031-36978-0_5. URL https://doi.org/10.1007/978-3-031-36978-0_5.

Thierry Coquand and Henrik Persson. Gröbner bases in type theory. In *Types for proofs and programs (Irsee, 1998)*, volume 1657 of *Lecture Notes in Comput. Sci.*, pages 33–46. Springer, Berlin, 1999. ISBN 3-540-66537-4. doi: 10.1007/3-540-48167-2_3. URL https://doi.org/10.1007/3-540-48167-2_3.

Tor H. Gulliksen. A theory of length for Noetherian modules. *J. Pure Appl. Algebra*, 3: 159–170, 1973. ISSN 0022-4049,1873-1376. doi: 10.1016/0022-4049(73)90030-3. URL https://doi.org/10.1016/0022-4049(73)90030-3.

Scott Huddleston and Jerry Shurman. Transfinite Chomp. In *More games of no chance (Berkeley, CA, 2000)*, volume 42 of *Math. Sci. Res. Inst. Publ.*, pages 183–212. Cambridge Univ. Press, Cambridge, 2002. ISBN 0-521-80832-4.

References II

Carl Jacobsson and Clas Löfwall. Standard bases for general coefficient rings and a new constructive proof of Hilbert's basis theorem. *J. Symbolic Comput.*, 12(3):337–371, 1991. ISSN 0747-7171,1095-855X. doi: 10.1016/S0747-7171(08)80154-X. URL https://doi.org/10.1016/S0747-7171(08)80154-X.

Gregor Kemper and Ngo Viet Trung. Krull dimension and monomial orders. *J. Algebra*, 399:782–800, 2014. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2013.10.005. URL https://doi.org/10.1016/j.jalgebra.2013.10.005.

Gregor Kemper and Ihsen Yengui. Valuative dimension and monomial orders. *J. Algebra*, 557:278–288, 2020. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2020.04.017. URL https://doi.org/10.1016/j.jalgebra.2020.04.017.

Ryota Kuroki. A quantitative Hilbert's basis theorem and the constructive Krull dimension, 2025. URL https://arxiv.org/abs/2509.00363.

Henri Lombardi. Dimension de Krull, Nullstellensätze et évaluation dynamique. *Math. Z.*, 242(1):23–46, 2002. ISSN 0025-5874,1432-1823. doi: 10.1007/s002090100305. URL https://doi.org/10.1007/s002090100305.

Henri Lombardi and Claude Quitté. *Commutative algebra: constructive methods*, volume 20 of *Algebra and Applications*. Springer, Dordrecht, revised edition, 2015. ISBN 978-94-017-9944-7. doi: 10.1007/978-94-017-9944-7. Finite projective modules, Translated from the French by Tania K. Roblot.

References III

Theodore Motzkin. The Euclidean algorithm. *Bull. Amer. Math. Soc.*, 55:1142–1146, 1949. ISSN 0002-9904. doi: 10.1090/S0002-9904-1949-09344-8. URL `https://doi.org/10.1090/S0002-9904-1949-09344-8`.

Fred Richman. Constructive aspects of Noetherian rings. *Proc. Amer. Math. Soc.*, 44: 436–441, 1974. ISSN 0002-9939,1088-6826. doi: 10.2307/2040452. URL `https://doi.org/10.2307/2040452`.

Abraham Seidenberg. What is Noetherian? *Rend. Sem. Mat. Fis. Milano*, 44:55–61, 1974. ISSN 0370-7377. doi: 10.1007/BF02925651. URL `https://doi.org/10.1007/BF02925651`.